THERMAL INSTABILITY OF A SHEAR FLOW IN AN ANNULAR CHANNEL

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1. Formulation of the Problem. An incompressible reactive fluid located between two coaxial cylinders is brought into motion by internal and external cylinders rotating uniformly with the angular velocities ω_1 and ω_2. In the shear flow – which is assumed to be steady and laminar – temperature increases due to an exothermic reaction and viscous drag. The problem is to determine the thermal stability of the fluid in relation to its properties, the dimensions of the channel, the boundary conditions, and the parameters of the flow. We will ignore the boundary conditions at the ends of the cylinders.

The model being examined here can be encountered in different production processes. In particular, it accurately models the determination of the viscosity of explosives in Couette–Gatchek viscometers [1].

The model is represented mathematically in the form

$$\frac{\partial T}{\partial t} = \frac{\varkappa}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right) + \frac{\nu}{c}\left[r\frac{\partial}{\partial r}\left(\frac{v}{r}\right)\right]^2 + \varphi(T); \tag{1.1}$$

$$\frac{d}{dr}\left(\frac{1}{r}\frac{d}{dr}(rv)\right) = 0, \quad T(r, 0) = T_0, \; R_1 \leqslant r \leqslant R_2; \tag{1.2}$$

$$\frac{\partial T(R_1, t)}{\partial r} + h_1(T(R_1, t) - T_0) = 0, \quad \frac{\partial T(R_2, t)}{\partial r} + h_2(T(R_2, t) - T_0) = 0; \tag{1.3}$$

$$v\ (r = R_1) = \omega_1 R_1, \; v\ (r = R_2) = \omega_2 R_2, \tag{1.4}$$

where t is time; r is radius; T is the temperature of the fluid; κ is diffusivity; ν is kinematic viscosity; c is heat capacity; v is the tangential velocity of the flow; R_1 and R_2 are the radii of the internal and external cylinders; h_1 and h_2 are constants; $\varphi(T)$ is a function satisfying the conditions $\varphi(T_0) > 0$, $\partial\varphi(T)/\partial T > 0$. Without loss of generality, we assume that the rate of heat release obeys the Arrhenius law

$$\varphi(T) = Qz(\rho c)^{-1}\exp(-E(RT)^{-1}).$$

Here, Q is the thermal effect of the reactions; z is a pre-exponential multiplier; E is activation energy; ρ is density; R is the universal gas constant.

After integrating (1.2) with allowance for (1.4) and introducing the dimensionless parameters $\Theta = E(T - T_0)R^{-1}T_0^{-2}$, $T = tt_a^{-1}$ $r = rr_a^{-1}$, $\bar{R}_1 = R_1 r_a^{-1}$, $\bar{R}_2 = R_2 r_a^{-1}$, $\beta = RT_0E^{-1}$, $\bar{h}_1 = h_1 r_a$, $\bar{h}_2 = h_2 r_a$, $\varphi(\Theta)$ exp $(\Theta(1 + \beta\Theta)^{-1})$, $\bar{q} = 4\nu\rho(\omega_2 - \omega_1)^2\bar{R}_1^4\bar{R}_2^4(\bar{R}_2^2 - \bar{R}_1^2)^{-2}(Qz)^{-1}$ $(t_a = c\rho RT_0^2 \times (EQz)^{-1}\exp(\bar{E}(RT_0)^{-1})$, $r_a = (\kappa t_a)^{0.5})$ we reduce system (1.1)-(1.3) to the form

$$\frac{\partial\Theta}{\partial\tau} = \frac{1}{\bar{r}}\frac{\partial}{\partial\bar{r}}\left(\bar{r}\frac{\partial\Theta}{\partial\bar{r}}\right) + \Delta\left(\bar{q}\exp(\beta^{-1})\bar{r}^{-4} + a_0\right) + \sum_{n=1}^{\infty} a_n\Theta^n = G(\Theta, \mu, \Delta); \tag{1.5}$$

$$\Theta(\bar{r}, 0) = 0, \frac{\partial\Theta(\bar{R}_1, \tau)}{\partial\bar{r}} + \bar{h}_1\Theta(\bar{R}_1, \tau) = 0, \frac{\partial\Theta(\bar{R}_2, \tau)}{\partial\bar{r}} + \bar{h}_2\Theta(\bar{R}_2, \tau) = 0, \tag{1.6}$$

where μ is a parameter from an interval containing zero; Δ is a parameter found from the relation $G(0, 0, \Delta) = \bar{q}\exp(\beta^{-1})\bar{r}^{-4} + a_0$; $a_n = 1/n! \times (\delta^n/\delta\Theta^n)\varphi(\Theta)|_{\Theta=0}$ are coefficients of the expansion of the function $\varphi(\Theta)$ into a series in powers of Θ.

In accordance with the central set theorem [2], infinite-dimensional problem (1.5)-(1.6) can be reduced to the space of finite dimensions without any loss of information re-

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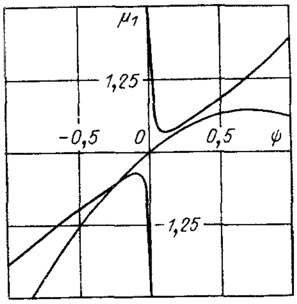

Fig. 1

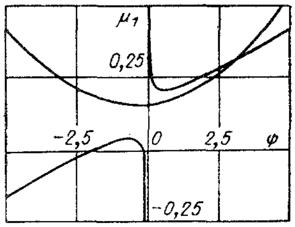

Fig. 2

garding the stability of its solutions. The simplest method of reducing the dimensionality of an infinite-dimensional problem is the approach in [3, 4]. The latter is based on the use of projections of the solutions of the problem on the space of eigenfunctions. These projections are used in conjunction with the Fredholm alternative theorem.

In accordance with this method, we introduce the auxiliary operators

$$\frac{\partial\Theta}{\partial\tau}=\frac{1}{\bar{r}}\frac{\partial}{\partial\bar{r}}\left(\bar{r}\frac{\partial\Theta}{\partial\bar{r}}\right)+\sum_{i=1}^{\infty}a_i\Theta^i=F(\Theta,\mu)=G(\Theta,\mu,0); \tag{1.7}$$

$$\frac{\partial\Theta}{\partial\tau}=\frac{1}{\bar{r}}\frac{\partial}{\partial\bar{r}}\left(\bar{r}\frac{\partial\Theta}{\partial\bar{r}}\right)+a_1\Theta=\frac{\partial F(0,0)}{\partial\Theta}\Theta+\mu\frac{\partial^2F(0,0)}{\partial\Theta\,\partial\mu}\Theta, \tag{1.8}$$

for which conditions (1.6) are valid.

To attract solutions of problem (1.5)-(1.6), with parameters distributed in R^∞, to the space R^2, we construct the space of eigenfunctions of the generatrix of operator (1.8) with allowance for (1.6) and we determine the stability of the zeroth solution. We then project the solutions of bifuraction problem (1.7) and problem (1.5) (containing the parameter $\Delta \neq 0$ which eliminates the bifuraction) on this space and we determine the stability of the solution of these problems.

Within the framework of the formulation of the problem presented above, it is useful to point out that the assumption that the flow is steady and laminar is valid if the condition of stability of a flow between rotating cylinders is satisfied [4]. When there is a small difference $|\omega_2 - \omega_1|$, this condition is given by the inequality [4]

$$\nu>\bar{R}_1^2\bar{R}_2^2\,|\,\omega_2-\omega_1\,|\,(\pi^2(\bar{R}_2^2-\bar{R}_1^2))^{-1}(\ln(\bar{R}_2R_1^{-1}))^2.$$

2. Zeroth Solution. The analysis of the stability of zeroth solution (1.5)-(1.6) reduces to a Sturm-Liouville problem involving determination of the eigenvalues of operator (1.8). The spectrum of operator (1.8) consists only of two discrete eigenvalues $\sigma_n=a_1-\lambda_n^2$, where λ_n (n = 1, 2,...) are positive roots of the equation

$$\det\begin{vmatrix}\bar{h}_1I_0(\lambda\bar{R}_1)-I_1(\lambda\bar{R}_1) & \bar{h}_1N_0(\lambda\bar{R}_1)-N_1(\lambda\bar{R}_1)\\ \bar{h}_2I_0(\lambda\bar{R}_2)-I_1(\lambda\bar{R}_2) & \bar{h}_2N_0(\lambda\bar{R}_2)-N_1(\lambda\bar{R}_2)\end{vmatrix}=0$$

($I_n(\lambda\bar{r})$, $N_n(\lambda\bar{r})$ are Bessel and Neumann functions of order n).

The parameter μ will henceforth be represented in the form of a function of the maximum eigenvalue σ_1, $\mu = \mu(\sigma_1)$. However, we will do this in such a way that, in accordance with the definition, the interval of its values will contain zero. In the zeroth case, the solution is stable if $\mu = a_1 - \lambda_1^2 < 0$.

All of the eigevalues σ_n of operator (1.8) are paired, with each eigenvalue corresponding to two eigenvectors:

$$y_{1n}=I_0(\lambda_n\bar{r}),\ y_{2n}=c_nN_0(\lambda_n\bar{r})$$

$$(c_n=(I_1(\lambda_n\bar{R}_1)-\bar{h}_1I_0(\lambda_n\bar{R}_1))(\bar{h}_1N_0(\lambda_n\bar{R}_1)-N_1(\lambda_n\bar{R}_1))^{-1}).$$

The algebraic and geometric multiplicities of the eigenvalues σ_n are the same, so that the vectors y_{ij} ($i = 1, 2, j = 1, 2, \ldots$) are independent and can be used to construct a biorthogonal system on ($\bar{R}_1$, $\bar{R}_2$). It follows from this that any solution of (1.5), (1.7) reduces to the form of an expansion in eigenvectors of operator (1.8).

3. Bifurcative Solution. The introduction of Gram-Charlier transforms reduces the vectors y_{ij} to a biorthogonal system, so that the eigenvalue σ_1 will correspond to the orthogonal vectors

$$\bar{y}_{11} = y_{11},\ \bar{y}_{21} = y_{21} - \langle y_{21},\ \bar{y}_{11}\bar{r}\rangle\ \|\bar{y}_{11}\|\bar{y}_{11}, \tag{3.1}$$

where $\langle\bar{y}_{ij},\ \bar{y}_{nm}\rangle$ is the scalar product of the vectors $\bar{y}_{ij}$, $\bar{y}_{nm}$, $\|\bar{y}_{ij}\| = \langle\bar{y}_{ij},\ \bar{y}_{ij}\bar{r}\rangle$.

The space of the vectors $\bar{y}_{ij}$ ($i = 1, 2, j = 1, 2, \ldots$) is a Hilbert space with the scalar product

$$\langle(\bar{y}_{1i},\ \bar{y}_{2j}),\ (\bar{y}_{1n},\ \bar{y}_{2m})\rangle = \langle\bar{y}_{1i},\ \bar{y}^*_{1n}\rangle + \langle\bar{y}_{2j},\ \bar{y}^*_{2m}\rangle$$

($\bar{y}^*_{1i}$, $\bar{y}^*_{2i}$ are the vectors conjugate to $\bar{y}_{1i}$, $\bar{y}_{2i}$). Since vectors $\bar{y}_{ij}$ are orthogonal on ($\bar{R}_1$, $\bar{R}_2$) with the weight $\bar{r}$, we choose the following as the vectors conjugate to $\bar{y}_{11}$, $\bar{y}_{21}$

$$\bar{y}^*_{i1} = \bar{y}_{i1}\bar{r}\left(\sum_{i=1}^{2}\|\bar{y}_{1i}\|\right)^{-1} \quad (i = 1, 2).$$

The subspace stretched over the vectors $\bar{y}_{11}$, $\bar{y}_{21}$ is orthogonal to the remaining part of the Hilbert space. Thus, any solution $\Theta = \Theta(\mu)$ of operator (1.7) can always be expanded into a part belonging to the two-dimensional null space of operator (1.8) and a part which is orthogonal to $\bar{y}^*_{11}$, $\bar{y}^*_{21}$.

The solution of (1.7), with conditions (1.6), can be found in the form of a power series

$$\begin{vmatrix}\Theta\\ \mu\end{vmatrix} = \sum_{n=1}^{\infty}\frac{\varepsilon^n}{n!}\begin{vmatrix}\Theta_n\\ \mu_n\end{vmatrix} \tag{3.2}$$

($\varepsilon = \langle(\Theta,\ \Theta),\ (\bar{y}_{11},\ \bar{y}_{21})\rangle$ is amplitude).

Insertion of (3.2) into (1.7) and identification of the terms with independent powers of ε leads to

$$\frac{\partial F\ (0,0)}{\partial\Theta}\Theta_1 = 0; \tag{3.3}$$

$$\frac{\partial F\ (0,0)}{\partial\Theta}\Theta_2 + 2\mu_1\frac{\partial^2 F\ (0,0)}{\partial\Theta\ \partial\mu}\Theta_1 + \frac{\partial^2 F\ (0,0)}{\partial\Theta^2}\Theta_1^2 = 0 \tag{3.4}$$

and equations with higher powers of ε. It follows directly from (3.3) that the solution can be any linear combination $\Theta_1 = \bar{y}_{11} + \psi\bar{y}_{21}$ (ψ is a parameter of the problem which is subject to determination). Equation (3.4) can be solved only in the case when the conditions $\langle\partial F(0,\ 0)/\partial\Theta\Theta_2,\ \bar{y}^*_{k1}\rangle = 0$, are satisfied for $k = 1, 2$, these conditions being based on the Fredholm alternative theorem. It follows from this that

$$2\mu_1\langle\partial^2 F(0,0)/\partial\Theta\ \partial\mu\Theta_1,\ \bar{y}^*_{k1}\rangle + \langle\partial^2 F(0,0)/\partial\Theta^2\Theta_1^2,\ \bar{y}^*_{k1}\rangle = 0 \quad (k = 1, 2). \tag{3.5}$$

The presence of two independent parameters μ_1 and ψ guarantees the existence of a solution to system (3.5). Insertion of expressions for Θ_1, $\bar{y}^*_{k1}$ ($k = 1, 2$) into these equations yields two equations of conic sections on the plane (μ_1, ψ):

$$g_1(\mu_1,\ \psi) = c_{11}\psi^2 + c_{12}\psi + c_{13}\mu_1\psi + c_{14}\mu_1 + c_{15} = 0; \tag{3.6}$$

$$g_2(\mu_1,\ \psi) = c_{21}\psi^2 + c_{22}\psi + c_{23}\mu_1\psi + c_{24}\mu_1 + c_{25} = 0, \tag{3.7}$$

where

$$c_{11} = 0{,}5\,\langle \partial^2 F(0,0)/\partial\Theta^2 \bar{y}_{21}^2,\ \bar{y}_{11}^*\rangle;\ c_{12} = \langle \partial^2 F(0,0)/\partial\Theta^2 \bar{y}_{11}\bar{y}_{21},\ \bar{y}_{11}^*\rangle;$$
$$c_{13} = \langle \partial^2 F(0,0)/\partial\Theta\partial\mu \bar{y}_{21},\ \bar{y}_{11}^*\rangle;\quad c_{14} = \langle \partial^2 F(0,0)/\partial\Theta\partial\mu \bar{y}_{11},\ \bar{y}_{11}^*\rangle;$$
$$c_{15} = 0{,}5\,\langle \partial^2 F(0,0)/\partial\Theta^2 \bar{y}_{11}^2,\ \bar{y}_{11}^*\rangle;\ c_{21} = 0{,}5\,\langle \partial^2 F(0,0)/\partial\Theta^2 \bar{y}_{21}^2,\ \bar{y}_{21}^*\rangle;$$
$$c_{22} = \langle \partial^2 F(0,0)/\partial\Theta^2 \bar{y}_{11}\bar{y}_{21},\ \bar{y}_{21}^*\rangle;\ c_{23} = \langle \partial^2 F(0,0)/\partial\Theta\partial\mu \bar{y}_{21},\ \bar{y}_{21}^*\rangle;$$
$$c_{24} = \langle \partial^2 F(0,0)/\partial\Theta\partial\mu \bar{y}_{11},\ \bar{y}_{21}^*\rangle;\ c_{25} = 0{,}5\,\langle \partial^2 F(0,0)/\partial\Theta^2 \bar{y}_{11}^2,\ \bar{y}_{21}^*\rangle.$$

By virtue of (3.1), $c_{13} = c_{24} = 0$, and if (3.6), (3.7) are not degenerate, then (3.6) will always be a parabola and (3.7) will always be a hyperbola. The points of intersection of curves (3.6), (3.7) $(\mu_1^{(n)}, \psi^{(n)})$ on the plane (μ_1, ψ) are solutions of (3.4). Depending on the sign of the discriminant of equivalent system (3.6)-(3.7) of the cubic equation

$$b_3\psi^3 + b_2\psi^2 + b_1\psi + b_0 = 0$$
$$\left(b_3 = 1,\ b_2 = \left(c_{12} - c_{14}c_{21}c_{23}^{-1}\right)c_{11}^{-1},\right. \tag{3.8}$$
$$\left.b_1 = \left(c_{15} - c_{14}c_{22}c_{23}^{-1}\right)c_{11}^{-1},\ b_0 = -\,c_{14}c_{25}c_{23}^{-1}c_{11}^{-1}\right)$$

system (3.6)-(3.7) will have either three real solutions or one real solution and two complex-conjugate solutions. If the discriminant is equal to zero, then two or three real solutions will coincide.

Numerous calculations performed at random points of the intervals $0.1 \le \bar{R}_1 \le 2.4$, $0.8 \le \bar{R}_2 \le 5$, $\beta = 0.02$, $1 \le \bar{h}_1 \le 2$, $-2 \le \bar{h}_2 \le 2$, $\bar{h}_1 = \bar{h}_2 \gg 1$, showed that nondegenerate system (3.6)-(3.7) always has one real solution (determining the point of steady equilibrium) and two complex solutions (in which periodic cycles appear). Thus, at $\bar{R}_1 = 1$, $\bar{R}_2 = 4.07$, $\beta = 0.02$, $\bar{h}_1 = \bar{h}_2 \gg 1$ system (3.6)-(3.7) has one real solution $(\mu_1^{(1)}, \psi(1)) = (-0.55, -0.24)$ (Fig. 1) and two complex solutions $(\mu_1^{(2)}, \psi^{(2)}) = (0.25 + 0.25i,\ 0.13 + 0.16i)$, $(\mu_1^{(3)}, \psi^{(3)}) = (0.25 - 0.25i,\ 0.13 - 0.16i)$. At $\bar{R}_1 = 1$, $\bar{R}_2 = 2.55$ and the same β, $\bar{h}_1$, $\bar{h}_2$, $(\mu_1^{(1)}, \psi^{(1)}) = (0.30,\ 2.94)$ (Fig. 2), $(\mu_1^{(2)}, \psi^{(2)}) = (0.16 - 0.016i,\ 0.36 - 0.83i)$, $(\mu_1^{(3)}, \psi^{(3)}) = (0.16 + 0.016i,\ 0.36 + 0.83i)$.

The stability of solutions (1.6)-(1.7) should be analyzed at each point of intersection of curves (3.6), (3.7) $(\mu_1^{(n)}, \psi^{(n)})$, n = 1-3. To do this, it is necessary to represent the relations $g_i(\mu_1, \psi)$ (i = 1, 2) in the form of functions of the parameter μ. Combining (3.2), (3.6), and (3.7) and using the normalization condition $\varepsilon = 1$, we can write system (3.6)-(3.7) in the form

$$\bar{g}_i(\mu) = \mu^2\left(c_{i1}\psi^2\mu_1^{-2} + c_{i2}\psi\mu_1^{-2} + c_{i3}\psi\mu_1^{-1} + c_{i4}\mu_1^{-1} + c_{i5}\mu_1^{-2}\right) = 0 \quad (i = 1, 2). \tag{3.9}$$

In accordance with the first approximation [5] of the Lyapunov stability theorem, the solution (1.6)-(1.7) will be stable if the real parts of the eigenvalues of the Jacobian matrix $I = |a_{ij}|$, where $a_{11} = \partial\bar{g}_1(\mu)/\partial\mu_1^{-1}$, $a_{12} = \partial\bar{g}_1(\mu)/\partial(\psi\mu_1^{-1})$, $a_{21} = \partial\bar{g}_2(\mu)/\partial\mu_1^{-1}$, $a_{22} = \partial\bar{g}_2(\mu)/\partial(\psi\mu_1^{-1})$, are negative. Considering that at each point $(\mu_1^{(n)}, \psi^{(n)})$ with small μ we have $I = \mu^2 \det I(\mu_1^{(n)}, \psi^{(n)}) + O|\mu|^3$, we can write the stability conditions as follows: for steady equilibrium

$$\max\left(\mu s_1^{(n)},\ \mu s_2^{(n)}\right) < 0,\ \det I\left(\mu_1^{(n)}, \psi^{(n)}\right) > 0; \tag{3.10}$$

for periodic cycles

$$\max\left(\mu\,\mathrm{Re}\,s_1^{(n)},\ \mu\,\mathrm{Re}\,s_2^{(n)}\right) < 0,$$
$$\left|\mathrm{Re}\left(a_{11}^{(n)} + a_{22}^{(n)}\right)\right| > \left|\left(\alpha_n^2 + \beta_n^2\right)^{0{,}25}\cos 0{,}5\,\mathrm{arctg}\,\alpha_n^{-1}\beta_n\right|. \tag{3.11}$$

Here, $s_1^{(n)}$, $s_2^{(n)}$ are eigenvalues of the matrix

$$I\left(\mu_1^{(n)}, \psi^{(n)}\right);\ a_{ij}^{(n)} = a_{ij}\left(\mu_1^{(n)}, \psi^{(n)}\right);$$

$$\alpha_n = \left(\mathrm{Re}\left(a_{11}^{(n)} - a_{22}^{(n)}\right)\right)^2 - \left(\mathrm{Im}\left(a_{11}^{(n)} - a_{22}^{(n)}\right)\right)^2 + 4\,\mathrm{Re}\, a_{12}^{(n)}\,\mathrm{Re}\, a_{21}^{(n)} - 4\,\mathrm{Im}\, a_{12}^{(n)}\,\mathrm{Im}\, a_{21}^{(n)};\ \beta_n = 2\,\mathrm{Re}\left(a_{11}^{(n)} - a_{22}^{(n)}\right)\mathrm{Im}\left(a_{11}^{(n)} - a_{22}^{(n)}\right) + 4\,\mathrm{Re}\, a_{12}^{(n)}\,\mathrm{Im}\, a_{21}^{(n)} + 4\,\mathrm{Re}\, a_{21}^{(n)}\,\mathrm{Im}\, a_{12}^{(n)}.$$

Conditions (3.10) are valid if and only if curves (3.6)-(3.7) intersect transversely on the plane (μ_1, ψ), i.e., if det $I_0 \neq 0$, where

$$I_0 = \begin{vmatrix} \partial g_1(\mu_1, \psi)/\partial\mu_1 & \partial g_1(\mu_1, \psi)/\partial\psi \\ \partial g_2(\mu_1, \psi)/\partial\mu_1 & \partial g_2(\mu_1, \psi)/\partial\psi \end{vmatrix}.$$

When Eq. (3.8) has only one real root, the intersection is always exact (if curves (3.6) and (3.7) touch, then we already have two real roots). If curve (3.6) and (3.7) have a common tangent at the point of intersection, then it is necessary to go back and reexamine the equation which follows (3.4). If in (3.11) max $(\mu\ \mathrm{Re}\ s_1^{(n)},\ \mu\ \mathrm{Re}\ s_2^{(n)}) = 0$, and the zero is simple, then we see the limit cycle - Hopf bifurcation [2].

An analysis of the stability of the thermal state of a fluid in an annular channel with $\bar{R}_1 = 1$, $\bar{R}_2 = 4.07$, $\beta = 0.02$, $\bar{h}_1 = \bar{h}_2 \gg 1$ shows that $(s_1^{(1)}, s_2^{(1)}) = (28.80, -0.25)$, $(s_1^{(2)}, s_2^{(2)}) = (4.72 - 20.25i, -20.08 - 4.74i)$, $(s_1^{(3)}, s_2^{(3)}) = (4.72 + 20.25i, -20.08 + 4.74i)$. Thus, the equilibrium state and the oscillatory cycles are unstable on either side of the critical point $\mu = 0$. At $\bar{R}_1 = 1$, $\bar{R}_2 = 2.55$, $\beta = 0.02$, $\bar{h}_1 = \bar{h}_2 \gg 1$, calculations yield $(s_1^{(1)}, s_2^{(1)}) = (10^{-2}, -3.10\cdot 10^{-2})$, $(s_1^{(2)}, s_2^{(2)}) = (-4.71\cdot 10^{-4} + 2.38\cdot 10^{-3}i, -4.14\cdot 10^{-2} - 6.04\cdot 10^{-4}i)$, $(s_1^{(3)}, s_2^{(3)}) = (-4.71\cdot 10^{-4} - 2.38\cdot 10^{-3}i, -4.14\cdot 10^{-2} + 6.04\cdot 10^{-4}i)$. Thus, the equilibrium state is unstable for any $\mu > 0$ and $\mu < 0$, while the oscillatory states are stable when $\mu = 4a_1 - \lambda_1^2 > 0$.

4. Isolated Solutions. The operator $G(\Theta, \mu, \Delta)$ contains the parameter $\Delta \neq 0$, which eliminates bifurcation at the point $(\mu, \Theta) = (0, 0)$. As a result, a solution which branches at this point decomposes into isolated solutions. At $\Delta = 0$, the solution $\Theta = 0$ of the equation $G(\Theta, \mu, 0) = 0$ becomes unstable when μ passes through zero. In accordance with the Hopf theorem [2], this is equivalent to the point $(\mu, \Theta) = (0, 0)$ being a double point.

The inequality $\langle \partial G(0, 0, 0)/\partial\Delta, \bar{y}^*_{k1}\rangle \neq 0$ $(k = 1, 2)$ and the implicit function theorem guarantee the existence of the solution $G(\Theta, \mu, \Delta) = 0$ relative to $\Delta = \Delta(\mu, \varepsilon)$, which we seek in the form of a series in powers of μ and ε.

We obtain the following system of equations from double differentiation of $G(\Theta, \mu, \Delta)$ with respect to μ, ε at the point $(\mu, \varepsilon) = (0, 0)$ and use of the identity $\partial G(\Theta, \mu, \Delta(\mu, 0))/\partial\varepsilon = 0$, which follows from the definition of a double point

$$\frac{\partial G(0,0,0)}{\partial\Theta}\frac{\partial^2\Theta}{\partial\varepsilon^2} + \frac{\partial^2 G(0,0,0)}{\partial\Theta^2}\Theta_1^2 + \frac{\partial G(0,0,0)}{\partial\Delta}\frac{\partial^2\Delta}{\partial\varepsilon^2} = 0; \tag{4.1}$$

$$\frac{\partial G(0,0,0)}{\partial\Theta}\frac{\partial^2\Theta}{\partial\mu\,\partial\varepsilon} + \frac{\partial^2 G(0,0,0)}{\partial\mu\,\partial\varepsilon}\Theta_1 + \frac{\partial G(0,0,0)}{\partial\Delta}\frac{\partial^2\Delta}{\partial\mu\,\partial\varepsilon} = 0. \tag{4.2}$$

This system can be solved only when the following conditions are satisfied for k = 1, 2

$$\left\langle \frac{\partial^2\Theta}{\partial\varepsilon^2}, \bar{y}^*_{k1}\right\rangle = \left\langle \frac{\partial^2\Theta}{\partial\mu\,\partial\varepsilon}, \bar{y}^*_{k1}\right\rangle = 0.$$

The last relations, together with Eqs. (4.1)-(4.2), determine the first two nontrivial terms in the expansion of the function in powers of μ, ε:

$$\Delta(\mu, \varepsilon) = -\frac{1}{2}\left[\frac{\langle \partial^2 G(0,0,0)/\partial\Theta^2\Theta_1^2, \bar{y}^*_{k1}\rangle}{\langle \partial G(0,0,0)/\partial\Delta, \bar{y}^*_{k1}\rangle}\varepsilon^2 + 2\frac{\langle \partial^2 G(0,0,0)/\partial\Theta\,\partial\mu\Theta_1, \bar{y}^*_{k1}\rangle}{\langle \partial G(0,0,0)/\partial\Delta, \bar{y}^*_{k1}\rangle}\varepsilon\mu\right] \quad (k = 1,2). \tag{4.3}$$

Equations (4.3) make it possible to find the solutions of (1.5), (1.6) on the plane (μ, ε).

Substitution of the expressions for Θ_1, $\bar{y}^*_{k1}$ (k = 1, 2) into (4.3) and use of the normalization condition $\varepsilon = 1$ yield

$$g_1(\mu_1, \psi) + \Delta \langle \partial G(0, 0, 0)/\partial\Delta, \bar{y}^*_{11}\rangle = 0; \tag{4.4}$$

$$g_2(\mu_1, \psi) + \Delta \langle \partial G(0, 0, 0)/\partial\Delta, \bar{y}^*_{21}\rangle = 0, \tag{4.5}$$

where $g_i(\mu_1, \psi)$ are determined from (3.6)-(3.7).

Changing over from (4.4)-(4.5) to the equivalent cubic equation leads to (3.8). Here, the coefficients b_2, b_3 are the same as in (3.8), while b_0 and b_1 have the form

$$b_0 = -\left(c_{25} + \Delta \langle \partial G(0, 0, 0)/\partial\Delta, \bar{y}^*_{21}\rangle\right) c_{23}^{-1} c_{11}^{-1},$$
$$b_1 = \left(c_{15} + \Delta \langle \partial G(0, 0, 0)/\partial\Delta, \bar{y}^*_{11}\rangle - c_{14}c_{22}c_{23}^{-1}\right) c_{11}^{-1}.$$

The subsequent analysis of the stability of the solutions of (1.5), (1.6) is similar to the procedure described in Part 3 for the bifurcative solution. The distribution of the real and complex solutions of system (4.4)-(4.5) is the same as for (3.6)-(3.7), although they need not coincide.

Calculations performed for $\bar{R}_1 = 1$, $\bar{R}_2 = 2.55$, $\beta = 0.02$, $\bar{h}_1 = \bar{h}_2 \gg 1$, $\partial G(0, 0, 0)/\partial\Delta = a_0$, $\bar{q} = 0$, gave $(\mu_1^{(1)}, \psi^{(1)}) = (3.10, 0.64)$, $(s_1^{(1)}, s_2^{(1)}) = (0.14, -0,76)$, $(\mu_1^{(2)}, \psi^{(2)}) = (0.22 + 0.74i, 1.51 + 15.53i)$, $(s_1^{(2)}, s_2^{(2)}) = (-9.14\cdot10^{-3} - 3.40\cdot10^{-2}i, -1.46 + 0.13i)$; the solution which is complex-conjugate to $(\mu_1^{(2)}, \psi^{(2)})$ is not presented here. It follows from this that the stationary state is unstable for any μ, while the periodic cycles are stable when $\mu > 0$.

Subsequent calculations performed with initial data that was the same as the above except for $\partial G(0, 0, 0)/\partial\Delta = a_0 + \bar{q}\exp(\beta^{-1})\bar{r}^{-4}$, $\bar{q}\exp(\beta^{-1}) = 1$, gave $(\mu_1^{(1)}, \psi^{(1)}) = (3.38, 1.20)$, $(s_1^{(1)}, s_2^{(1)}) = (0.15, -0.82)$, $(\mu_1^{(2)}, \psi^{(2)}) = (0.21 + 0.66i, 1.22 + 16.20i)$, $(s_1^{(2)}, s_2^{(2)}) = (-2.66\cdot10^{-3} - 3.08\cdot10^{-2}i, -1.60 + 9.60\cdot10^{-2}i)$. The distribution of stability is the same as in the previous case. Further calculations showed that with unchanged $\bar{R}_1 = 1$, $\bar{R}_2 = 2.55$, $\beta = 0,02$, $\bar{h}_1 = \bar{h}_2 \gg 1$, an increase in flow velocity up to $\bar{q} = \infty$ does not qualitatively affect the pattern of stability distribution. This can be attributed to the fact that at $\bar{h}_1 = \bar{h}_2 \gg 1$, the increase in the temperature of the fluid due to viscous drag is accompanied by intensive heat removal through the surface $\bar{R}_1$, $\bar{R}_2$, where the temperature is kept equal to zero ($\Theta = 0$). At the same time, it must be pointed out that the power of the attractor corresponding to the unstable equilibrium state increases with an increase in flow velocity.

The conclusion that there is no qualitative change in the pattern of stability distribution with a change in flow velocity of course applies only to the specific parameters used in the calculations. For other problem parameters, a change in flow velocity may indeed lead to a qualitative change in stability distribution. Thus, at $\bar{R}_1 = 1$, $\bar{R}_2 = 2.30$, $\bar{h}_1 = \bar{h}_2 \gg 1$, system (4.4)-(4.5) becomes degenerate and (4.3) has only one solution $\mu\varepsilon - (0.17 - 0.34\beta)\varepsilon^2 - 3.18 - 0.84\bar{q}\exp(\beta^{-1}) = 0$. It follows from this that, for certain values of β, there is a threshold value of $\bar{q}$ at which μ changes sign.

Proceeding on the basis of the above analysis of the solutions of system (1.5)-(1.6), it is tempting to conclude the following. A flow of an initially uniformly heated fluid in which temperature is increasing due to exothermic reactions and viscous drag can be in one of three thermal states: an unstable state of thermal equilibrium and two periodic cycles which may be either stable or unstable. With the problem formulated as it was here (without allowance for combustion of the reactants during the reactions), a limiting cycle is possible. There are cylinder diameters (such that one of the two eigenvectors corresponding to each paired eigenvalue σ_n (n = 1, 2,...) of the generating operator (1.8) is equal to zero) for which the fluid will have only one steady state of thermal equilibrium. This state may be either stable or unstable.

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MODELING THERMAL PROCESSES IN THE SHOCK LOADING OF POROUS MATERIALS

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Shock loading of porous materials within the pressure range 1.5-3 GPa makes it possible to obtain monolithic products [1-3]. Higher loading pressures (30-100 GPa) are used to study the equations of state of different substances under critical conditions [4]. It was shown in [5] that shock loading is characterized by highly nonequilibrium thermal processes. This high level of disequilibrium is due both to the sudden compression of the substance behind the shock wave (SW) and to the effect of powerful heat flows on the surface of particles of the substance. At the low pressures characteristic of powder compaction methods, thermal nonequilibrium is manifest in fusion of the particle surface [2, 3].

An increase in pressure or porosity in the pressing operation leads to an increase in the proportion of the substance that is converted to the liquid state and to a further intensification of heat release at the boundaries of particles. In this case, it is necessary to recognize the existence of a region of the substance in which the temperature exceeds the boiling point at normal pressure. Two layers can in turn be discerned within this region. In the first layer, internal energy is greater than the energy of vaporization, while in the second layer the former is less than the latter. When pressure is relieved, the molten layer - in which the acquired internal heat energy exceeds the energy necessary for vaporization - changes to the vapor state. Bulk boiling should be expected to occur in the second layer. The state of the substance behind the shock front actually depends on the ratio t^*/τ (where t^* is the time of arrival of the unloading wave, $\tau = R^2/a$ is the characteristic time of establishment of thermal equilibrium in an individual particle of radius R, and a is the diffusivity of the particle). At $t^*/\tau \ll 1$, there is not sufficient time for the particle to be heated uniformly, and vaporization and boiling may take place in the superheated material after the arrival of the unloading wave. When $t^*/\tau \gg 1$, an equilibrium temperature is established in the particle. Here, the state of the substance after arrival of the unloading wave is determined by the p-v diagram.

When the energy of the SW is low, the substance remains in the solid state. If the adiabatic curve corresponding to unloading passes through a two-phase region, then the substance is dispersed. When the curve passes above the critical point, the material vaporizes.

Thus, by varying the energy expended in shock compression and the parameter t^*/τ, it is possible to use the shock loading of porous materials to obtain different final states: monolithic solids, finely dispersed powders with a developed porous surface, ultradispersed particles, or a dense plasma.

We will examine the dynamic loading of powdered metal by a plane shock wave. The internal energy of the substance ε behind the front of the SW can be determined from the Hugoniot curve $\varepsilon = (p + p_0)(1/\rho_{00} - 1/\rho)/2$ and represented in the form of the sum: $\varepsilon = \varepsilon_g + \varepsilon_d + \varepsilon_t$. Here, ρ_{00} and ρ are the initial and final densities ahead of and behind the shock front; p is pressure; ε_g, ε_d, and ε_t are the fractions of energy expended by the SW on

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